FINITE SIMPLE LABELED GRAPH C^*-ALGEBRAS OF CANTOR MINIMAL SUBSHIFTS

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Abstract. It is now well known that a simple graph C^*-algebra $C^*(E)$ of a directed graph E is either AF or purely infinite. In this paper, we address the question of whether this is the case for labeled graph C^*-algebras recently introduced by Bates and Pask as one of the generalizations of graph C^*-algebras, and show that there exists a family of simple unital labeled graph C^*-algebras which are neither AF nor purely infinite. Actually these algebras are shown to be isomorphic to crossed products $C(X)\times_T\mathbb{Z}$ where the dynamical systems (X,T) are Cantor minimal subshifts. Then it is an immediate consequence of well known results about this type of crossed products that each labeled graph C^*-algebra in the family obtained here is an $A\mathbb{T}$ algebra with real rank zero and has $\mathbb{Z}$ as its K_1-group.

1. Introduction

With the motivation to provide a common framework for studying the ultragraph C^*-algebras ([30, 31]) and the shift space C^*-algebras (see [7, 8, 26] among others), Bates and Pask [3] introduced the C^*-algebras associated to labeled graphs (more precisely, labeled spaces). Graph C^*-algebras (see [2, 5, 24, 25, 29] among many others) and Exel-Laca algebras [13] are ultragraph C^*-algebras and all these algebras are defined as universal objects generated by partial isometries and projections satisfying certain relations determined by graphs (for graph C^*-algebras), ultragraphs (for ultragraph C^*-algebras), and infinite matrices (for Exel-Laca algebras). In a similar but more complicated manner, a labeled graph C^*-algebra $C^*(E,\mathcal{L},\mathcal{B})$ is also defined as a C^*-algebra generated by partial isometries $\{s_a : a\in\mathcal{A}\}$ and projections $\{p_A : A\in\mathcal{B}\}$, where $\mathcal{A}$ is an alphabet onto which a *labeling map* $\mathcal{L}: E^1\to\mathcal{A}$ is given from the edge set E^1 of the directed graph E, and $\mathcal{B}$, an *accommodating set*, is a set of vertex subsets $A\subset E^0$ satisfying certain conditions. The family of these generators is assumed to obey a set of rules regulated by the triple $(E,\mathcal{L},\mathcal{B})$ called a *labeled space* and moreover it should be universal in the sense that any C^*-algebra generated by a family of partial isometries and projections satisfying the same rules must be a quotient algebra of $C^*(E,\mathcal{L},\mathcal{B})$. The universal property allows the group $\mathbb{T}$ to act on $C^*(E,\mathcal{L},\mathcal{B})$ in a canonical way, and this action γ (called the *gauge action*) plays an important role throughout the study of generalizations

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of the Cuntz-Krieger algebras. The Cuntz-Krieger algebras [10] (and the Cuntz algebras [9]) are the C^*-algebras of finite graphs from which many generalizations have emerged in various ways including the C^*-algebras of higher-rank graphs whose study started in [23].

Simplicity and pure infiniteness results for labeled graph C^*-algebras are obtained in [4], and particularly it is shown that there exists a purely infinite simple labeled graph C^*-algebra which is not stably isomorphic to any graph C^*-algebras. Thus we can say that the class of simple labeled graph C^*-algebras is strictly larger than that of simple graph C^*-algebras. As is shown in [31], every simple ultragraph C^*-algebra is either AF or purely infinite, whereas we know from [27] that among higher rank graph C^*-algebras there exist simple C^*-algebras which are neither AF nor purely infinite, more specifically there exist such simple C^*-algebras which are stably isomorphic to irrational rotation algebras or Bunce-Deddens algebras. These examples of finite (but non-AF) simple C^*-algebras associated to higher rank graphs raise a natural question of whether there exist labeled graph C^*-algebras that are simple finite but non-AF. The purpose of this paper is to answer this question positively by providing a family of such simple labeled graph C^*-algebras. The C^*-algebras in this family are $A\mathbb{T}$-algebras (limit circle algebras) with traces that are isomorphic to crossed products $C(X) \times_T \mathbb{Z}$ of Cantor minimal systems (X, T), where the compact metric spaces X are subshifts over finite alphabets.

A dynamical system (X, T) consists of a compact metrizable space X and a transformation $T : X \to X$ which is a homeomorphism. This determines a C^*-dynamical system $(C(X), \mathbb{Z}, T)$ where $T(f) := f \circ T^{-1}$, $f \in C(X)$ and thus gives rise to the crossed product $C(X) \times_T \mathbb{Z}$. If two dynamical systems (X_i, T_i), $i = 1, 2$, are topologically conjugate, namely if there is an homeomorphism $\phi : X_1 \to X_2$ satisfying $T_2(\phi(x)) = \phi(T_1(x))$ for all $x \in X$, then it is rather obvious that the crossed products are isomorphic. As a consequence of the Markov-Kakutani fixed point theorem, one can show that there exists a Borel probability measure m on X which is T-invariant in the sense that $m \circ T^{-1} = m$ (for example, see [11, Theorem VIII. 3.1]). If there exists a unique T-invariant measure, we call (X, T) *uniquely ergodic*. If X is the only non-empty closed T-invariant subspace of X, the system (X, T) is said to be *minimal*, and as is well known, a dynamical system (X, T) is minimal if and only if each T-orbit $\{T^i x : i \in \mathbb{Z}\}$, $x \in X$, is dense in X. A Cantor space is characterized as a compact metrizable totally disconnected space with no isolated points, and a dynamical system (X, T) on a Cantor space X is called a *Cantor system*. The family of Cantor minimal systems is important for the study of whole minimal dynamical systems in view of the fact that every minimal system is a factor of a Cantor minimal system (see [15, Section 1]).

For a dynamical system (X, T) on an infinite space X, the crossed product $C(X) \times_T \mathbb{Z}$ is well known to be simple exactly when the system (X, T) is minimal. In particular, if (X, T) is a minimal dynamical system on a Cantor space X, this simple crossed product turns out to be an $A\mathbb{T}$-algebra, an inductive limit of finite direct sums of matrix algebras over $\mathbb{C}$ or $C(\mathbb{T})$ (for example, see [11, Chapter VIII]). It should be noted here that these simple crossed products $C(X) \times_T \mathbb{Z}$ of Cantor minimal systems are never AF since their K_1 groups are all equal to $\mathbb{Z}$, hence nonzero ([16, Theorem 1.4]).

For a finite alphabet $\mathcal{A}$ ($|\mathcal{A}| \geq 2$), the set $\mathcal{A}^{\mathbb{Z}}$ of all two-sided infinite sequences becomes a compact metrizable space in the product topology and forms a dynamical system $(\mathcal{A}^{\mathbb{Z}}, T)$ together with the shift transformation T given by $T(\omega)_i := \omega_{i+1}$, $\omega \in \mathcal{A}^{\mathbb{Z}}$, $i \in \mathbb{Z}$. If $X \subset \mathcal{A}^{\mathbb{Z}}$ is a T-invariant closed subspace, we call the dynamical system (X, T) a *subshift* of $(\mathcal{A}^{\mathbb{Z}}, T)$. For a sequence $\omega \in \mathcal{A}^{\mathbb{Z}}$, let $\mathcal{O}_\omega$ denote the the closure of the T-orbit of ω. Then, as is well known, the subshift $(\mathcal{O}_\omega, T)$ becomes a Cantor minimal system whenever ω is an almost periodic and aperiodic sequence.

In order to explain how we form a labeled space from a Cantor minimal subshift $(\mathcal{O}_\omega, T)$, let $E_{\mathbb{Z}}$ be the directed graph with the vertex set $\{v_n : n \in \mathbb{Z}\}$ and the edge set $\{e_n : n \in \mathbb{Z}\}$, where each e_n is an arrow from v_n to v_{n+1}, $n \in \mathbb{Z}$. Then we consider a labeling map $\mathcal{L}_\omega$ on the graph $E_{\mathbb{Z}}$ which assigns to an edge e_n a letter ω_n for each $n \in \mathbb{Z}$. In this way we obtain a labeled graph C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$, where $\overline{\mathcal{E}}_{\mathbb{Z}}$ is the smallest set amongst the normal accommodating sets. Then we first show that these unital labeled graph algebras $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ are all simple and have traces. In the simple crossed product $C(\mathcal{O}_\omega) \times_T \mathbb{Z}$, we then find a family of partial isometries and projections satisfying the same relations required for the canonical generators of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$, which proves from universal property of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ that there exists an isomorphism of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ to the crossed product $C(\mathcal{O}_\omega) \times_T \mathbb{Z}$. Our results can be summarized as follows:

Theorem 1.1. *(Theorem 3.7 and Theorem 3.10) Let $\mathcal{A}$ be a finite alphabet with $|\mathcal{A}| \geq 2$, and let $\omega \in \mathcal{A}^{\mathbb{Z}}$ be a sequence such that the subshift $(\mathcal{O}_\omega, T)$ is a Cantor minimal system. If $\mathcal{L}_\omega$ is a labeling map on the graph $E_{\mathbb{Z}}$ by the sequence ω, the labeled graph C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is a non-AF simple unital C^*-algebra. Moreover there is an isomorphism*

$$\pi : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}}) \to C(\mathcal{O}_\omega) \times_T \mathbb{Z}$$

such that the restriction of π onto the fixed point algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ of the gauge action γ is an isomorphism onto $C(\mathcal{O}_\omega)$.

The crossed products $C(X) \times_T \mathbb{Z}$ of Cantor minimal systems have been studied intensively (especially in [15, 16]). Perhaps one important result from the works, in our viewpoint, would be the fact that the crossed products $C(\mathcal{O}_\omega) \times_T \mathbb{Z}$ can be completely classified by their ordered K_0-groups with distinguished oder units ([15, Theorem 2.1]). Also from the above theorem and [16, Theorem 1.4] we know that the labeled graph C^*-algebras $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ are $A\mathbb{T}$-algebras with $K_1(C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})) = \mathbb{Z}$, hence they are not AF.

Finally, regarding the question of abundance of those Cantor minimal subshift systems, we notice a well known fact that (X, T) is topologically conjugate to a two-sided subshift if and only if it is expansive, and also from [12, Theorem 1] that this is the case if a Cantor system (X, T) has a finite rank K and $K \geq 2$ while odometer systems are the systems of rank one (we refer the reader to [12] for definitions and properties of this sort of systems).

2. Preliminaries

2.1. **Labeled spaces.** We will follow notational conventions of [24] for graph C^*-algebras and of [4, 1] for labeled spaces and their C^*-algebras. A *directed graph* $E=(E^0,E^1,r,s)$ consists of a countable vertex set E^0, a countable edge set E^1, and the range, source maps $r,\ s:E^1\to E^0$. If $v\in E^0$ emits (receives, respectively) no edges it is called a *sink* (*source*, respectively). Throughout this paper, we assume that *graphs have no sinks and no sources.*

E^n denotes the set of all finite paths $\lambda=\lambda_1\cdots\lambda_n$ of *length* n ($|\lambda|=n$), ($\lambda_i\in E^1,\ r(\lambda_i)=s(\lambda_{i+1}), 1\le i\le n-1$). We write $E^{\le n}$ and $E^{\ge n}$ for the sets $\cup_{i=1}^n E^i$ and $\cup_{i=n}^\infty E^i$, respectively. The range and source maps, r and s, naturally extend to all finite paths $E^{\ge 0}$, where $r(v)=s(v)=v$ for $v\in E^0$. If a sequence of edges $\lambda_i\in E^1(i\ge 1)$ satisfies $r(\lambda_i)=s(\lambda_{i+1})$, one has an infinite path $\lambda_1\lambda_2\lambda_3\cdots$ with the source vertex $s(\lambda_1\lambda_2\lambda_3\cdots):=s(\lambda_1)$. By E^∞ we denote the set of all infinite paths.

A *labeled graph* $(E,\mathcal{L})$ over a countable alphabet $\mathcal{A}$ consists of a directed graph E and a *labeling map* $\mathcal{L}:E^1\to\mathcal{A}$. For $\lambda=\lambda_1\cdots\lambda_n\in E^{\ge1}$, we call $\mathcal{L}(\lambda):=\mathcal{L}(\lambda_1)\cdots\mathcal{L}(\lambda_n)$ a (*labeled*) *path*. Similarly one can define an infinite labeled path $\mathcal{L}(\lambda)$ for $\lambda\in E^\infty$. A labeled graph $(E,\mathcal{L})$ is said to have a *repeatable path* β if $\beta^n:=\beta\cdots\beta(\text{repeated } n\text{-times})\in\mathcal{L}(E^{\ge1})$ for all $n\ge1$. The *range* $r(\alpha)$ of a labeled path $\alpha\in\mathcal{L}(E^{\ge1})$ is defined to be a vertex subset of E^0:

$$r(\alpha)=\{r(\lambda)\,:\,\lambda\in E^{\ge1},\,\mathcal{L}(\lambda)=\alpha\},$$

and the *source* $s(\alpha)$ of α is defined similarly. The *relative range of* $\alpha\in\mathcal{L}(E^{\ge1})$ *with respect to* $A\subset 2^{E^0}$ is defined to be

$$r(A,\alpha)=\{r(\lambda)\,:\,\lambda\in E^{\ge1},\ \mathcal{L}(\lambda)=\alpha,\ s(\lambda)\in A\}.$$

For notational convenience, we use a symbol ϵ such that $r(\epsilon)=E^0$, $r(A,\epsilon)=A$ for all $A\subset E^0$, and $\alpha=\epsilon\alpha=\alpha\epsilon$ for all $\alpha\in\mathcal{L}(E^{\ge1})$, and write

$$\mathcal{L}^{\#}(E):=\mathcal{L}(E^{\ge1})\cup\{\epsilon\}.$$

We denote the subpath $\alpha_i\cdots\alpha_j$ of $\alpha=\alpha_1\alpha_2\cdots\alpha_{|\alpha|}\in\mathcal{L}(E^{\ge1})$ by $\alpha_{[i,j]}$ for $1\le i\le j\le|\alpha|$. A subpath of the form $\alpha_{[1,j]}$ is called an *initial path* of α. The symbol ϵ is regarded as an initial (and terminal) path of every path.

Let $\mathcal{B}\subset 2^{E^0}$ be a collection of subsets of E^0. If $r(A,\alpha)\in\mathcal{B}$ for all $A\in\mathcal{B}$ and $\alpha\in\mathcal{L}(E^{\ge1})$, $\mathcal{B}$ is said to be *closed under relative ranges* for $(E,\mathcal{L})$. We call $\mathcal{B}$ an *accommodating set* for $(E,\mathcal{L})$ if it is closed under relative ranges, finite intersections and unions and contains $r(\alpha)$ for all $\alpha\in\mathcal{L}(E^{\ge1})$. The triple $(E,\mathcal{L},\mathcal{B})$ is called a *labeled space* when $\mathcal{B}$ is accommodating for $(E,\mathcal{L})$.

For $A,B\in 2^{E^0}$ and $n\ge1$, let

$$AE^n=\{\lambda\in E^n\,:\,s(\lambda)\in A\},\quad E^nB=\{\lambda\in E^n\,:\,r(\lambda)\in B\}.$$

We write E^nv for $E^n\{v\}$ and vE^n for $\{v\}E^n$, and will use notations like $AE^{\ge k}$ and vE^∞ which should have obvious meaning. A labeled space $(E,\mathcal{L},\mathcal{B})$ is said to be *set-finite* (*receiver set-finite*, respectively) if for every $A\in\mathcal{B}$ and $l\ge1$ the set $\mathcal{L}(AE^l)$ ($\mathcal{L}(E^lA)$, respectively) is finite. A labeled space $(E,\mathcal{L},\mathcal{B})$ is *finite* if there are only finitely many labels.

In this paper, we will always assume that labeled spaces $(E,\mathcal{L},\mathcal{B})$ are *weakly left-resolving*, namely

$$r(A,\alpha)\cap r(B,\alpha)=r(A\cap B,\alpha)$$

for all $A,B\in\mathcal{B}$ and $\alpha\in\mathcal{L}(E^{\geq 1})$. $(E,\mathcal{L},\mathcal{B})$ is *left-resolving* if $\mathcal{L}:r^{-1}(v)\to\mathcal{A}$ is injective for each $v\in E^0$. Left-resolving labeled spaces are weakly left-resolving.

For each $l\geq 1$, the following relation on E^0,

$$v\sim_l w \text{ if and only if } \mathcal{L}(E^{\leq l}v)=\mathcal{L}(E^{\leq l}w)$$

is actually an equivalence relation, and the equivalence class $[v]_l$ of $v\in E^0$ is called a *generalized vertex*. If $k>l$, $[v]_k\subseteq[v]_l$ is obvious and $[v]_l=\cup_{i=1}^m[v_i]_{l+1}$ for some vertices $v_1,\dots,v_m\in[v]_l$ ([4, Proposition 2.4]).

Notation 2.1. Given a labeled graph $(E,\mathcal{L})$, $\overline{\mathcal{E}}$ denotes the smallest *normal* accommodating set, that is the smallest one among the accommodating sets which are closed under relative complements.

Proposition 2.2. ([4, Remark 2.1 and Proposition 2.4.(ii)], [20, Proposition 2.3]) *Let $(E,\mathcal{L})$ be a labeled graph (E has no sinks or sources). Then*

$$\overline{\mathcal{E}}=\{\cup_{i=1}^n[v_i]_l\,:\,v_i\in E^0,\ l,n\geq 1\}.$$

2.2. **Labeled graph C^*-algebras.** Here we review the labeled graph C^*-algebras which are associated to set-finite, receiver set-finite, and weakly left-resolving labeled spaces (whose underlying graphs have no sinks or sources) although our results are concerning only about finite left-resolving spaces.

Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space such that $\overline{\mathcal{E}}\subset\mathcal{B}$. Recall from [1, Definition 2.1] that a *representation* of $(E,\mathcal{L},\mathcal{B})$ is a collection of projections $\{p_A : A\in\mathcal{B}\}$ and partial isometries $\{s_a : a\in\mathcal{A}\}$ such that for $A,B\in\mathcal{B}$ and $a,b\in\mathcal{A}$,

(i) $p_\emptyset=0$, $p_Ap_B=p_{A\cap B}$, and $p_{A\cup B}=p_A+p_B-p_{A\cap B}$,
(ii) $p_As_a=s_ap_{r(A,a)}$,
(iii) $s_a^*s_a=p_{r(a)}$ and $s_a^*s_b=0$ unless $a=b$,
(iv) for each $A\in\mathcal{B}$,

$$p_A=\sum_{a\in\mathcal{L}(AE^1)}s_ap_{r(A,a)}s_a^*. \tag{1}$$

It follows from (iv) that $p_A=\sum_{\alpha\in\mathcal{L}(AE^n)}s_\alpha p_{r(A,\alpha)}s_\alpha^*$ for $n\geq 1$. By $C^*(p_A,s_a)$ we denote the C^*-algebra generated by $\{s_a,p_A : a\in\mathcal{A},\ A\in\mathcal{B}\}$.

Remark 2.3. Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space such that $\overline{\mathcal{E}}\subset\mathcal{B}$.

(i) There exists a C^*-algebra generated by a universal representation $\{s_a,p_A\}$ of $(E,\mathcal{L},\mathcal{B})$ (see the proof of [3, Theorem 4.5]). If $\{s_a,p_A\}$ is a universal representation of $(E,\mathcal{L},\mathcal{B})$, we call $C^*(s_a,p_A)$, denoted $C^*(E,\mathcal{L},\mathcal{B})$, the *labeled graph C^*-algebra* of $(E,\mathcal{L},\mathcal{B})$. Note that $s_a\neq 0$ and $p_A\neq 0$ for $a\in\mathcal{A}$

and $A \in \mathcal{B}$, $A \neq \emptyset$, and that $s_\alpha p_A s_\beta^* \neq 0$ if and only if $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$. By definition of representation and [3, Lemma 4.4], it follows that

$$C^*(E, \mathcal{L}, \mathcal{B}) = \overline{span}\{s_\alpha p_A s_\beta^* \,:\, \alpha,\, \beta \in \mathcal{L}^{\#}(E),\ A \in \mathcal{B}\}, \tag{2}$$

where s_ϵ is regarded as the unit of the multiplier algebra of $C^*(E, \mathcal{L}, \mathcal{B})$.

(ii) Universal property of $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$ defines the *gauge action* $\gamma : \mathbb{T} \to Aut(C^*(E, \mathcal{L}, \mathcal{B}))$ such that for $a \in \mathcal{L}(E^1)$, $A \in \mathcal{B}$, and $z \in \mathbb{T}$,

$$\gamma_z(s_a) = z s_a \ \text{ and } \ \gamma_z(p_A) = p_A.$$

(iii) The fixed point algebra of γ is an AF algebra such that

$$C^*(E, \mathcal{L}, \mathcal{B})^\gamma = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : |\alpha| = |\beta|,\ A \in \mathcal{B}\} \tag{3}$$

Moreover, since $\mathbb{T}$ is a compact group, there exists a faithful conditional expectation

$$\Psi : C^*(E, \mathcal{L}, \mathcal{B}) \to C^*(E, \mathcal{L}, \mathcal{B})^\gamma.$$

Recall [4, 18] that for a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$, a path $\alpha \in \mathcal{L}([v]_l E^{\geq 1})$ is *agreeable* for a generalized vertex $[v]_l$ if $\alpha = \beta^k \beta'$ for some $\beta \in \mathcal{L}([v]_l E^{\leq l})$ and its initial path β', and $k \geq 1$. A labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ is said to be *disagreeable* if every $[v]_l$, $l \geq 1$, $v \in E^0$, is disagreeable in the sense that there is an $N \geq 1$ such that for all $n \geq N$ there is a path $\alpha \in \mathcal{L}([v]_l E^{\geq n})$ which is not *agreeable*.

Remark 2.4. If $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable, every representation $\{s_a, p_A\}$ such that $p_A \neq 0$ for all non-empty set $A \in \overline{\mathcal{E}}$ gives rise to a C^*-algebra $C^*(s_a, p_A)$ isomorphic to $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ ([4, Theorem 5.5] and [19, Corollary 2.5]). A labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable if there is no repeatable paths in $(E, \mathcal{L})$ ([20, Proposition 4.12]).

K-theory of labeled graph C^*-algebras was obtained in [1]. Let $(E, \mathcal{L}, \mathcal{B})$ be a normal labeled space. Since we assume that E has no sink vertices ($E^0_{\text{sink}} = \emptyset$), the set $\mathcal{B}_J$ given in (2) of [1] coincides with $\mathcal{B}$, and by [1, Theorem 4.4] the linear map $(1 - \Phi) : \operatorname{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}\} \to \operatorname{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}\}$ given by

$$(1 - \Phi)(\chi_A) = \chi_A - \sum_{a \in \mathcal{L}(AE^1)} \chi_{r(A,a)}, \quad A \in \mathcal{B} \tag{4}$$

determines the K-groups of $C^*(E, \mathcal{L}, \mathcal{B})$ as follows:

$$K_0(C^*(E, \mathcal{L}, \mathcal{B})) \cong \operatorname{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}\}/\operatorname{Im}(1 - \Phi) \tag{5}$$

$$K_1(C^*(E, \mathcal{L}, \mathcal{B})) \cong \ker(1 - \Phi). \tag{6}$$

In (5), the isomorphism is given by $[p_A]_0 \mapsto \chi_A + \operatorname{Im}(1 - \Phi)$ for $A \in \mathcal{B}$.

2.3. **Cantor minimal systems that are subshifts.** A (topological) dynamical system (X, T) consists of a compact metrizable space X and a homeomorphism T on X. By Krylov-Bogolyubov Theorem, a dynamical system (X, T) admits a Borel probability measure m which is T-invariant, that is $m(T^{-1}(E)) = m(E)$ for all Borel sets E. If there exists exactly one T-invariant probability measure, we say that the system (X, T) is *uniquely ergodic*. We will focus on the Cantor systems

(X, T) that are subshifts, and here we briefly review definitions and basic properties of such Cantor systems.

For an alphabet $\mathcal{A}$ ($|\mathcal{A}| \geq 2$), a *word* (or *block*) over $\mathcal{A}$ is a finite sequence $b = b_1 \cdots b_k$ of symbols (or letters) b_i's in $\mathcal{A}$ of length $|b| := k \geq 1$. By $\mathcal{A}^+$, we denote the set of all *words*. Let ϵ be the empty word of length zero and let $\mathcal{A}^* := \mathcal{A}^+ \cup \{\epsilon\}$. The set

$$\mathcal{A}^{\mathbb{Z}} := \{\omega = \cdots \omega_{-1}\omega_0\omega_1 \cdots : \omega_i \in \mathcal{A}\}$$

of all two-sided infinite sequences on $\mathcal{A}$, endowed with the product topology of the discrete topology on $\mathcal{A}$, is a totally disconnected compact metrizable space. Actually the *cylinder sets*

$${}_t[b] := \{\omega \in \mathcal{A}^{\mathbb{Z}} : \omega_{[t,t+|b|-1]} = b\},$$

$b \in \mathcal{A}^+$, $t \in \mathbb{Z}$, are clopen and form a base for the topology, where $\omega_{[t_1,t_2]}$ denotes the block $\omega_{t_1} \cdots \omega_{t_2}$ ($t_1 \leq t_2$). Thus the characteristic functions $\chi_{{}_t[b]}$ are continuous for all $b \in \mathcal{A}^+$, $t \in \mathbb{Z}$. If $b = \omega_{[t_1,t_2]}$ holds for $b \in \mathcal{A}^+$ and $\omega \in \mathcal{A}^{\mathbb{Z}} \cup \mathcal{A}^+$, b is called a *factor* of ω. For $\omega \in \mathcal{A}^{\mathbb{Z}}$ (or $\mathcal{A}^{\mathbb{N}}$), the set of all factors of ω is denoted by

$$L_\omega = \{\omega_{[t_1,t_2]} : t_1 \leq t_2\}.$$

For convenience, we will use the following notation:

$$[.b] := {}_0[b], \quad [b.] := {}_{-|b|}[b], \quad [b.c] := {}_{-|b|}[bc]$$

for words $b, c \in \mathcal{A}^+$.

The *shift* transform $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by

$$(Tx)_k = x_{k+1}, \;\; k \in \mathbb{Z},$$

is a homeomorphism. A *subshift* on $\mathcal{A}$ is a (topological) dynamical system (X, T) which consists of a T-invariant closed subset $X \subset \mathcal{A}^{\mathbb{Z}}$ and the restriction $T|_X$ which we denote by T again. If we consider the shift transform T on the space $\mathcal{A}^{\mathbb{N}}$ of one-sided infinite sequences, it is a continuous transform (but not a homeomorphism).

For $\omega \in \mathcal{A}^{\mathbb{Z}}$, the closure of the orbit of ω is denoted by

$$\mathcal{O}_\omega := \overline{\{T^i(\omega) : i \in \mathbb{Z}\}} \;\subset \mathcal{A}^{\mathbb{Z}}.$$

A dynamical system (X, T) is *minimal* if every orbit is dense in X, namely $\mathcal{O}_x = X$ for all $x \in X$. It is well known that a subshift $(\mathcal{O}_\omega, T)$ is minimal if and only if ω is *almost periodic* (or *uniformly recurrent*) in the sense that each factor of ω occurs with bounded gaps.

We provide examples of subshifts that are Cantor minimal systems:

Example 2.5. (**Generalized-Morse sequences**) ([21]) Let $\mathcal{A} = \{0, 1\}$. For a one-sided sequence $x \in \mathcal{A}^{\mathbb{N}}$, let $\mathscr{O}_x := \{\omega \in \mathcal{A}^{\mathbb{Z}} : L_\omega \subset L_x\}$. Note that each block $b \in \mathcal{A}^+$ defines a block $\tilde{b}$, called the *mirror image* of b, such that $\tilde{b}_i = b_i + 1$ (mod 2). For $c = c_0 \cdots c_n \in \mathcal{A}^+$, the product $b \times c$ of b and c denotes the block (of length $|b| \times |c|$) obtained by putting $n+1$ copies of either b or $\tilde{b}$ next to each other according to the rule of choosing the ith copy as b if $c_i = 0$ and $\tilde{b}$ if $c_i = 1$. For example, if $b = 01$ and $c = 011$, then the product block $b \times c$ is equal to $b\tilde{b}\tilde{b} = 011010$.

Let $\{b^i := b^i_0 \cdots b^i_{|b^i|-1}\}_{i\geq 1} \subset \mathcal{A}^+$ be a sequence of blocks with length $|b^i| \geq 2$ such that $b^i_0 = 0$ for all $i \geq 0$. Since the product operation $\times$ is associative, one can consider a sequence of the form

$$x = b^0 \times b^1 \times b^2 \times \cdots \in \mathcal{A}^{\mathbb{N}}$$

which is called a (one-sided) *recurrent* sequence (see [21, Definition 7]). We call $x = b^0 \times b^1 \times b^2 \times \cdots \in \mathcal{A}^{\mathbb{N}}$ a *(generalized) one-sided Morse sequence* if it is non-periodic and

$$\sum_{i=0}^{\infty} \min(r_0(b^i), r_1(b^i)) = \infty,$$

where $r_a(b)$ is the *relative frequency of occurrence* of $a \in \mathcal{A}$ in $b \in \mathcal{A}^+$ (see [21, p.338]). If $x \in \mathcal{A}^{\mathbb{N}}$ is a non-periodic recurrent sequence, it is almost periodic, and there exists $\omega \in \mathscr{O}_x$ with $x = \omega_{[0,\infty)}$. Moreover, x is a one-sided Morse sequence if and only if $\mathcal{O}_\omega$ is minimal and uniquely ergodic, and if this is the case, then $\mathcal{O}_\omega = \mathscr{O}_x$.

By a *generalized Morse sequence*, we mean a two-sided sequence $\omega \in \mathcal{A}^{\mathbb{Z}}$ such that $x := \omega_{[0,\infty)}$ is a one-sided Morse sequence and $L_\omega = L_x$. (Note that the term a *two-sided generalized Morse sequence* used in [21] means a sequence $\omega \in \mathscr{O}_x$ for some one-sided Morse sequence x.)

The subshifts $(\mathcal{O}_\omega, T)$ for generalized Morse sequences ω are uniquely ergodic Cantor minimal systems.

Example 2.6. (**Substitution subshifts**) ([17]) Let $\mathcal{A}$ be a finite alphabet with $|\mathcal{A}| \geq 2$. A *substitution* on $\mathcal{A}$ is a map $\sigma : \mathcal{A} \to \mathcal{A}^+$. σ can be iterated to define maps $\sigma^k : \mathcal{A} \to \mathcal{A}^+$ for all positive integer k, and is called *primitive* if there exists $k \geq 1$ such that b appears in $\sigma^k(a)$ for all $a, b \in \mathcal{A}$. By the *language* L_σ of a substitution σ we mean the set of words that are factors of $\sigma^k(a)$ for some $k \geq 1$ and $a \in \mathcal{A}$. The subshift

$$X_\sigma := \{x \in \mathcal{A}^{\mathbb{Z}} \mid L_x \subset L_\sigma\},$$

associated to this language L_σ is called the *substitution subshift* defined by σ. If σ is primitive, it is known that the system (X_σ, T) is minimal and thus a Cantor minimal system.

A sequence $\omega \in \mathcal{A}^{\mathbb{Z}}$ is called a *fixed point* of σ if $\sigma(\omega) = \omega$, and it is known that for any primitive substitution σ, there is an $n \geq 1$ such that σ^n admits a fixed point ω in X_σ. Since σ^n and σ define the same dynamical system, we can only consider primitive substitutions σ with a fixed point $\omega \in X_\sigma$, and in this case, $X_\sigma = \mathcal{O}_\omega$ follows. To avoid the case where X_σ is finite, or equivalently ω is shift periodic, we also assume that σ is an *aperiodic* substitution (giving rise to the infinite system X_σ). Then the substitution subshifts $(X_\sigma, T) = (\mathcal{O}_\omega, T)$ are uniquely ergodic minimal Cantor systems.

Example 2.7. (**Thue-Morse sequence**) Let $\mathcal{A} = \{0, 1\}$ and $b^i := 01 \in \mathcal{A}^+$ for all $i \geq 0$. Then the recurrent sequence

$$x := b^0 \times b^1 \times b^2 \times \cdots = 01 \times b^1 \times \cdots = 0110 \times b^2 \times \cdots = 01101001 \times b^3 \times \cdots$$

is a one-sided Morse sequence called the *Thue-Morse sequence* and

$$\omega := x^{-1}.x = \cdots 10010110.011010011001 \cdots \in \mathscr{O}_x$$

is a generalized Morse sequence, where $x^{-1} := \cdots x_2x_1x_0$ is the sequence obtained by writing $x = x_0x_1\cdots$ in reverse order. In fact, ω is the sequence constructed from x in the proof of [21, Lemma 4], and it is well known [14] that ω is characterized as a sequence with no blocks of the form bbb_0 for any $b = b_0 \cdots b_{|b|-1} \in \mathcal{A}^+$. By Example 2.5, the subshift $(\mathcal{O}_\omega, T)$ is a uniquely ergodic Cantor minimal system.

On the other hand, this Thue Morse sequence ω is the fixed point of the primitive aperiodic substitution $\sigma : \mathcal{A} \to \mathcal{A}^+$ given by

$$\sigma(0) = 01 \quad \text{and} \quad \sigma(1) = 10,$$

so that the subshift $(\mathcal{O}_\omega, T)$ can also be viewed as a substitution subshift (X_σ, T).

3. Main results

Throughout this section, $E_\mathbb{Z}$ will denote the following graph:

$$\cdots \underset{v_{-4}}{\bullet} \xrightarrow{e_{-4}} \underset{v_{-3}}{\bullet} \xrightarrow{e_{-3}} \underset{v_{-2}}{\bullet} \xrightarrow{e_{-2}} \underset{v_{-1}}{\bullet} \xrightarrow{e_{-1}} \underset{v_0}{\bullet} \xrightarrow{e_0} \underset{v_1}{\bullet} \xrightarrow{e_1} \underset{v_2}{\bullet} \xrightarrow{e_2} \underset{v_3}{\bullet} \xrightarrow{e_3} \underset{v_4}{\bullet} \cdots .$$

Given a two-sided sequence $\omega = \cdots \omega_{-1}\omega_0\omega_1 \cdots \in \mathcal{A}^\mathbb{Z}$, we obtain a labeled graph $(E_\mathbb{Z}, \mathcal{L}_\omega)$ shown below

$$(E_\mathbb{Z}, \mathcal{L}_\omega) \ \cdots \underset{v_{-4}}{\bullet} \xrightarrow{\omega_{-4}} \underset{v_{-3}}{\bullet} \xrightarrow{\omega_{-3}} \underset{v_{-2}}{\bullet} \xrightarrow{\omega_{-2}} \underset{v_{-1}}{\bullet} \xrightarrow{\omega_{-1}} \underset{v_0}{\bullet} \xrightarrow{\omega_0} \underset{v_1}{\bullet} \xrightarrow{\omega_1} \underset{v_2}{\bullet} \xrightarrow{\omega_2} \underset{v_3}{\bullet} \xrightarrow{\omega_3} \underset{v_4}{\bullet} \cdots ,$$

where the labeling map $\mathcal{L}_\omega : E^1_\mathbb{Z} \to \mathcal{A}$ is given by $\mathcal{L}_\omega(e_n) = \omega_n$ for $e_n \in E^1_\mathbb{Z}$. Then we also have a labeled space $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ with the smallest accommodating set $\overline{\mathcal{E}}_\mathbb{Z}$ which is closed under relative complements.

Assumption. In this section, unless stated otherwise, $\mathcal{A}$ is a finite alphabet with $|\mathcal{A}| \geq 2$ and $\omega \in \mathcal{A}^\mathbb{Z}$ denotes an almost periodic sequence such that the subshift $(\mathcal{O}_\omega, T)$ is a Cantor minimal system.

3.1. **The fixed point algebra $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})^\gamma$ of the gauge action γ.** Let $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z}) = C^*(s_a, p_A)$ be the labeled graph C^*-algebra associated with the labeled space $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$. Since the labeled paths $\mathcal{L}_\omega(E^{\geq 1}_\mathbb{Z})$ are exactly the factors of the sequence ω, from now on we briefly denote the whole labeled paths by L_ω.

By (7), we know that the fixed point algebra of the gauge action γ is generated by elements of the form $s_\alpha p_A s^*_\beta$ ($|\alpha| = |\beta|$). But, in the case $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$, it is rather obvious that $s_\alpha p_A s^*_\beta \neq 0$, $|\alpha| = |\beta|$, only if $\alpha = \beta$ and $A \cap r(\alpha) \neq \emptyset$. Since $\mathcal{L}_\omega(E^l v)$ consists of a single path for each vertex v and $l \geq 1$, every generalized vertex $[v]_l$ is equal to the range $r(\alpha)$ for a path α with $\mathcal{L}_\omega(E^l v) = \{\alpha\}$. Hence, by

Proposition 2.2, we have

$$C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma = \overline{\operatorname{span}}\{s_\alpha p_{r(\beta\alpha)} s_\alpha^* : \alpha, \beta \in L_\omega\}. \tag{7}$$

Moreover $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ is easily seen to be a commutative C^*-algebra. For each $k \geq 1$, let

$$F_k := \operatorname{span}\{s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* : \alpha, \alpha' \in L_\omega, |\alpha| = |\alpha'| = k\}.$$

The (finitely many) elements $s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*$ in F_k are linearly independent and actually orthogonal to each other so that F_k is a finite dimensional subalgebra of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$. Moreover F_k is a subalgebra of F_{k+1} because

$$s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* = \sum_{b \in \mathcal{A}} s_{\alpha b} p_{r(\alpha'\alpha b)} s_{\alpha b}^* = \sum_{a,b \in \mathcal{A}} s_{\alpha b} p_{r(a\alpha'\alpha b)} s_{\alpha b}^*.$$

This gives rise to an inductive sequence $F_1 \xrightarrow{\iota_1} F_2 \xrightarrow{\iota_2} \cdots$ of finite dimensional C^*-algebras, where the connecting maps $\iota_k : F_k \to F_{k+1}$ are the inclusions for all $k \geq 1$, from which we obtain an AF algebra $\varinjlim F_k$.

Proposition 3.1. *For the labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$, we have*

$$C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma = \varinjlim F_k.$$

Proof. Since $F_k \subset C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ for all $k \geq 1$ and $\overline{\cup_k F_k} = \varinjlim F_k$, it is clear that $\varinjlim F_k \subset C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$. Thus it suffices to know that the algebra $\cup_k F_k$ is dense in $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ and then by (7) we only need to show that for $y := s_\alpha p_{r(\beta\alpha)} s_\alpha^*$, there is $k \geq 1$ such that $y \in F_k$. If $|\beta\alpha| = 2|\alpha|$, then $y \in F_k$ for $k = |\alpha|$. If $|\beta\alpha| > 2|\alpha|$, then

$$y = s_\alpha p_{r(\beta\alpha)} s_\alpha^* = \sum_{\nu \in \mathcal{L}_\omega(E^{|\beta|-|\alpha|})} s_{\alpha\nu} p_{r(\beta\alpha\nu)} s_{\alpha\nu}^* \in F_k$$

for $k = |\beta|$. If $|\beta\alpha| < 2|\alpha|$, we also have

$$y = s_\alpha p_{r(\beta\alpha)} s_\alpha^* = \sum_{\nu \in \mathcal{L}_\omega(E^{|\alpha|-|\beta|})} s_\alpha p_{r(\nu\beta\alpha)} s_\alpha^* \in F_k$$

for $k = |\alpha|$. ☐

Proposition 3.2. *There is a surjective isomorphism*

$$\rho : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \to C(\mathcal{O}_\omega) \tag{8}$$

such that $\rho(s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*) = \chi_{[\alpha'.\alpha]}$ for $s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* \in F_k$, $k \geq 1$.

Proof. Note that for each $k \geq 1$, the map $\rho_k : F_k \to C(\mathcal{O}_\omega)$ given by

$$\rho_k(s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*) = \chi_{[\alpha'.\alpha]}$$

is a $*$-homomorphism (we omit the proof) such that for $y = s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* \in F_k$,

$$\rho_k(y) = \rho_{k+1}(\iota_k(y)),$$

where $\iota_k : F_k \to F_{k+1}$ is the inclusion map. In fact, $\iota_k(y) = \sum_{a,b\in\mathcal{A}} s_{\alpha b} p_{r(a\alpha'\alpha b)} s_{\alpha b}^*$, so that

$$\rho_{k+1}(\iota_k(y)) = \rho_{k+1}\big(\sum_{a,b\in\mathcal{A}} s_{\alpha b} p_{r(a\alpha'\alpha b)} s_{\alpha b}^*\big) = \sum_{a,b\in\mathcal{A}} \chi_{[a\alpha'.\alpha b]}.$$

But $\sum_{a,b\in\mathcal{A}} \chi_{[a\alpha'.\alpha b]} = \chi_{[\alpha'.\alpha]}$ is obvious from $\cup_{a,b\in\mathcal{A}}[a\alpha'.\alpha b] = [\alpha'.\alpha]$. Hence, there exists a $*$-homomorphism $\rho : \varinjlim F_k \to C(\mathcal{O}_\omega)$ satisfying $\rho(y) = \rho_k(y)$ for all $y \in F_k$, $k \geq 1$. Since each ρ_k is injective, so is ρ.

Now we show that ρ is surjective to complete the proof. Let $\chi_{{}_t[\beta]} \in C(\mathcal{O}_\omega)$ for $t \in \mathbb{Z}$ and $\beta \in L_\omega$. Assuming $t > 0$, we can write $\chi_{{}_t[\beta]} = \sum\limits_{\alpha,\nu} \chi_{[\alpha.\nu\beta]}$, where the sum is taken over all α, ν with $|\nu| = t$ and $|\alpha| = |\nu\beta|$. Then for $k := |\beta| + t$, we have

$$\chi_{{}_t[\beta]} = \rho_k\big(\sum_{\alpha,\nu} s_\alpha p_{r(\alpha\nu\beta)} s_\alpha^*\big) \in \rho(F_k).$$

In the case $t \leq 0$, a similar argument shows that $\chi_{{}_t[\beta]} \in \rho(F_k)$ for some k. Thus ρ is surjective since $\mathrm{span}\{\chi_{{}_t[\beta]} : t \in \mathbb{Z},\, \beta \in L_\omega\}$ is a dense subalgebra of $C(\mathcal{O}_\omega)$. $\square$

Remark 3.3. It follows from general theory for dynamical systems that the systems $(\mathcal{O}_\omega, T)$ considered in this paper have always T-invariant ergodic probability measure (for example, see [11, Chapter VIII]). If m_ω is such a T-invariant ergodic measure, the unital commutative AF algebra $C(\mathcal{O}_\omega)$ of all continuous functions on $\mathcal{O}_\omega$ admits a (tracial) state

$$f \mapsto \int_{\mathcal{O}_\omega} f \mathrm{d}m_\omega : C(\mathcal{O}_\omega) \to \mathbb{C}$$

which we also write m_ω. Since m_ω is T-invariant, it easily follows that $m_\omega(\chi_{{}_t[b]}) = m_\omega(\chi_{{}_t[b]} \circ T) = m_\omega(\chi_{{}_{t+1}[b]})$, and hence

$$m_\omega(\chi_{{}_t[b]}) = m_\omega(\chi_{[.b]}) \tag{9}$$

holds for all $t \in \mathbb{Z}$ and $b \in L_\omega$.

Lemma 3.4. *Let $\rho : C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})^\gamma \to C(\mathcal{O}_\omega)$ be the isomorphism given in* (8)*. Then a T-invariant ergodic measure m_ω on $\mathcal{O}_\omega$ defines a tracial state*

$$\tau_0 := m_\omega \circ \rho : C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})^\gamma \to \mathbb{C}$$

on the fixed point algebra $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})^\gamma$ such that for $\alpha, \beta \in L_\omega$,

$$\tau_0(s_\alpha p_{r(\beta\alpha)} s_\alpha^*) = \tau_0(p_{r(\beta\alpha)}).$$

Proof. Note that $p_{r(\beta\alpha)} = \sum_\nu s_\nu p_{r(\beta\alpha\nu)} s_\nu^*$, where the sum is taken over the paths ν with $|\nu| = |\beta\alpha|$. We then have

$$\rho(p_{r(\beta\alpha)}) = \rho\big(\sum_{|\nu|=|\beta\alpha|} s_\nu p_{r(\beta\alpha\nu)} s_\nu^*\big) = \sum_{|\nu|=|\beta\alpha|} \chi_{[\beta\alpha.\nu]} = \chi_{\cup_\nu[\beta\alpha.\nu]} = \chi_{[\beta\alpha.]}.$$

Thus

$$\tau_0(p_{r(\beta\alpha)}) = m_\omega(\chi_{[\beta\alpha.]}).$$

On the other hand, if $|\beta\alpha| > 2|\alpha|$, $s_\alpha p_{r(\beta\alpha)} s_\alpha^* = \sum_{|\nu|=|\beta|-|\alpha|} s_{\alpha\nu} p_{r(\beta\alpha\nu)} s_{\alpha\nu}^*$ so that

$$\tau_0(s_\alpha p_{r(\beta\alpha)} s_\alpha^*) = m_\omega(\sum_{|\nu|=|\beta|-|\alpha|} \chi_{[\beta.\alpha\nu]}) = m_\omega(\chi_{[\beta.\alpha]}).$$

But the equality $m_\omega(\chi_{[\beta\alpha.]}) = m_\omega(\chi_{[\beta.\alpha]})$ follows from the fact that m_ω is T-invariant (see (9)). The case where $|\beta\alpha| \le 2|\alpha|$ can be done in a similar way. □

Lemma 3.5. *The labeled graph C^*-algebra $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ admits a tracial state*

$$\tau_0 \circ \Psi : C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z}) \to \mathbb{C},$$

where $\Psi : C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z}) \to C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})^\gamma$ is the conditional expectation onto the fixed point algebra of the gauge action.

Proof. To see that $\tau_0 \circ \Psi$ is a trace, we claim

$$\tau_0(\Psi(XY)) = \tau_0(\Psi(YX)) \tag{10}$$

for $X, Y \in \operatorname{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in L_\omega,\ A \in \overline{\mathcal{E}}_\mathbb{Z},\ A \subset r(\alpha) \cap r(\beta)\}$. Since the map $\tau_0 \circ \Psi$ is linear, we only need to show (10) for $X = s_\alpha p_A s_\beta^*$ and $Y = s_\mu p_B s_\nu^*$. But also by (1), it suffices to consider the case of $|\beta| = |\mu|$, so that $XY = \delta_{\beta,\mu} s_\alpha p_{A\cap B} s_\nu^*$. In this case if $|\alpha| \ne |\nu|$, then $\Psi(XY) = \Psi(YX) = 0$ follows immediately. Hence now let $|\alpha| = |\nu|$. If $\alpha \ne \nu$, it is easy to see that $XY = YX = 0$ and (10) holds. If $\alpha = \nu$, then $YX = s_\beta p_{B\cap A} s_\beta^*$ and $XY = s_\alpha p_{A\cap B} s_\alpha^*$, and by Lemma 3.4 we have

$$\tau_0(\Psi(XY)) = \tau_0(XY) = \tau_0(s_\alpha p_{A\cap B} s_\alpha^*) = \tau_0(p_{A\cap B}) = \tau_0(\Psi(YX)).$$

The fact that $\tau_0 \circ \Psi$ is a state comes from

$$(\tau_0 \circ \Psi)(1) = \tau_0\big(\sum_{a,b\in\mathcal{A}} s_b p_{r(ab)} s_b^*\big) = m_\omega(\sum_{a,b\in\mathcal{A}} \chi_{[a.b]}) = m_\omega(\chi_{\mathcal{O}_\omega}) = 1.$$

□

To prove the simplicity of the labeled graph C^*-algebra $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$, we need the following lemma which might be well known in the theory of dynamical systems, but we provide a proof here for the reader's convenience.

Lemma 3.6. *Let $\omega \in \mathcal{A}^\mathbb{Z}$ be a sequence which is almost periodic but not periodic. Then the labeled space $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ is disagreeable.*

Proof. It is enough to show that the labeled space has no repeatable paths (see Remark 2.4). For this, suppose there is a repeatable path α. We may assume that α has the smallest length. If $\beta \in L_\omega$, by the assumption that ω is almost periodic, there exists a $d \ge 1$ such that every block $\omega_{[t,t+d]}$, $t \in \mathbb{Z}$, has β as its factor. Thus any path $\alpha^k \in L_\omega$, k large enough, has β as a factor, so that $\alpha^k = \mu\beta\nu$ for some $\mu, \nu \in L_\omega \cup \{\epsilon\}$. In other words, every $\beta \in L_\omega$ must be of the form $\beta = \alpha''\alpha^l\alpha'$ for an initial path α' and terminal path α'' of α and $l \ge 0$.

Now we can apply this fact to the paths $\beta = \omega_{[0,n]}$, $n \ge 1$, to obtain that $\omega_{[0,\infty)}$ is of the form $\alpha''\alpha^\infty$. But then, considering the blocks of the form $\omega_{[-n,n]} \in L_\omega$

$(n \to \infty)$ we can easily see that $\omega = (\alpha)^\infty \alpha'.\alpha''(\alpha)^\infty$, where $\alpha = \alpha'\alpha''$. Thus ω is periodic, which is a contradiction. $\square$

Since we assume that a Cantor system $(\mathcal{O}_\omega, T)$ is minimal, or equivalently ω is almost periodic (and not periodic), the labeled space $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ considered in this section is always disagreeable by Lemma 3.6.

The following theorem shows that there exist simple labeled graph C^*-algebras that are not stably isomorphic to simple graph C^*-algebras.

Theorem 3.7. *Let $\mathcal{A}$ be a finite alphabet with $|\mathcal{A}| \geq 2$, and let $\omega \in \mathcal{A}^\mathbb{Z}$ be a sequence such that the subshift $(\mathcal{O}_\omega, T)$ is a Cantor minimal system. Then the labeled graph C^*-algebra $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ is a non-AF simple unital C^*-algebra with a tracial state τ which satisfies*

$$\tau(s_\alpha p_{r(\nu\alpha)} s_\beta^*) = \tau \circ \Psi(s_\alpha p_{r(\nu\alpha)} s_\beta^*) = \delta_{\alpha,\beta}\tau(p_{r(\nu\alpha)})$$

for labeled paths $\alpha, \beta, \nu \in \mathcal{L}_\omega(E_\mathbb{Z}^{\geq 1})$. Moreover if the system $(\mathcal{O}_\omega, T)$ is uniquely ergodic, $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ has a unique tracial state.

Proof. For the simplicity of $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z}) = C^*(p_A, s_a)$, we show that any nonzero homomorphism $\pi : C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z}) \to C^*(q_A, t_a)$ onto a C^*-algebra generated by $q_A := \pi(p_A)$, $t_a := \pi(s_a)$ for $A \in \overline{\mathcal{E}}_\mathbb{Z}$, $a \in \mathcal{A}$, is faithful. Since the labeled space $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ is disagreeable by Lemma 3.6, we see from [4, Theorem 5.5] that π is faithful whenever $\pi(p_{[v]_l}) \neq 0$ for all $v \in E^0$ and $l \geq 1$. Suppose on the contrary that

$$q_{[v]_m} = \pi(p_{[v]_m}) = 0$$

for some $[v]_m = r(\alpha)$ with $|\alpha| = m$. Since $\alpha \in L_\omega$ and ω is almost periodic, one finds a $d \geq 1$ such that for all $s \geq 0$,

$$T^{s+j}\omega \in [.\alpha]$$

for some $0 \leq j \leq d$. This means that if $\beta \in L_\omega$ is a block with length $|\beta| \geq d$, it must have α as a factor. Thus β must be of the form $\beta = \beta'\alpha\beta''$ for some $\beta', \beta'' \in \mathcal{L}_\omega^\sharp(E)\,(= L_\omega \cup \{\epsilon\})$. For β with $|\beta| \geq d$ we have $q_{r(\beta)} = 0$. In fact,

$$\begin{aligned}
q_{r(\beta)} &= q_{r(\beta'\alpha\beta'')} = q_{r(r(\beta'\alpha),\beta'')} \\
&= q_{r(r(\beta'\alpha),\beta''))} t_{\beta''}^* t_{\beta''} q_{r(r(\beta'\alpha),\beta'')} \\
&\sim t_{\beta''} q_{r(r(\beta'\alpha),\beta'')} t_{\beta''}^* \\
&\leq q_{r(\beta'\alpha)} \leq q_{r(\alpha)} \\
&= q_{[v]_m} = 0.
\end{aligned}$$

On the other hand, since π is a nonzero homomorphism, there exists a $\delta \in L_\omega$ with $q_{r(\delta)} = \pi(p_{r(\delta)}) \neq 0$. But then, with an $n > \max\{|\delta|, d\}$, we have

$$q_{r(\delta)} = \pi(p_{r(\delta)}) = \pi\Big(\sum_{|\delta\mu_i|=n} s_{\mu_i} p_{r(\delta\mu_i)} s_{\mu_i}^*\Big) = \sum_{|\delta\mu_i|=n} t_{\mu_i} q_{r(\delta\mu_i)} t_{\mu_i}^* = 0,$$

a contradiction, and $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ is simple.

With $\overline{\mathcal{E}}_{\mathbb{Z}}$ in place of $\mathcal{B}$ in (6) it is rather obvious that $\mathcal{N}=\emptyset$ and $\hat{\mathcal{B}}=\hat{\mathcal{B}}_J=\overline{\mathcal{E}}_{\mathbb{Z}}$. Since $\chi_A\in\ker(1-\Phi)$ if and only if $\chi_A=\sum_{a\in\mathcal{A}}\chi_{r(A,a)}$ (see (4)) which actually holds for $A=E^0_{\mathbb{Z}}$, we have $K_1(C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}}))=\ker(1-\Phi)\neq 0$. Thus $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ is not AF. (We will see later from Theorem 3.10 that $K_1(C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}}))=\mathbb{Z}$.)

If τ_0 is the tracial state of $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ induced by an ergodic measure of $(\mathcal{O}_\omega,T)$, the tracial state $\tau:=\tau_0\circ\Psi: C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})\to\mathbb{C}$ of Lemma 3.5 satisfies

$$\tau(s_\alpha p_{r(\nu\alpha)}s_\beta^*)=\delta_{\alpha,\beta}\tau(p_{r(\nu\alpha)}) \tag{11}$$

for $s_\alpha p_{r(\nu\alpha)}s_\beta^*\in C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$.

Now let $(\mathcal{O}_\omega,T)$ be uniquely ergodic and again let τ_0 be the tracial state on the fixed point algebra $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ and $\tau:=\tau_0\circ\Psi$ the extension of τ_0 to the whole labeled graph C^*-algebra as before. To show that τ is the unique tracial state on $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$, we claim that if τ' is a tracial state on $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$, then $\tau'\circ\Psi=\tau'$ holds, and that the state $\tau'\circ\rho^{-1}$ on $C(\mathcal{O}_\omega)$ is T-invariant. For the first claim, suppose $\tau'\circ\Psi\neq\tau'$. Then there exists an element $s_\alpha p_{r(\alpha)}s_\beta^*$ ($|\beta|<|\alpha|$) such that $\tau'(s_\alpha p_{r(\alpha)}s_\beta^*)\neq 0$. Since τ' is tracial, we have $0\neq\tau'(s_\alpha p_{r(\alpha)}s_\beta^*)=\tau'(s_\beta^* s_\alpha p_{r(\alpha)})$. Thus α must be of the form $\alpha=\beta\alpha'$ for some path α', and then $0\neq\tau'(s_\beta^*s_\alpha p_{r(\alpha)})=\tau'(s_{\alpha'}p_{r(\alpha)})$. Again the tracial property of τ' gives

$$0\neq\tau'(s_{\alpha'}p_{r(\alpha)})=\tau'(p_{r(\alpha)}s_{\alpha'})=\tau'(s_{\alpha'}p_{r(\alpha\alpha')})=\cdots=\tau'(s_{\alpha'}p_{(r(\alpha),(\alpha')^n)})$$

for all $n\geq 1$. But this means that the generalized vertex $[v]_l:=r(\alpha)$, $l=|\alpha|$, is not disagreeable emitting only agreeable paths, which is a contradiction to Lemma 3.6. To see that the state $\tau'\circ\rho^{-1}: C(\mathcal{O}_\omega)\to\mathbb{C}$ is T-invariant, let $\chi_{t[\beta]}\in C(\mathcal{O}_\omega)$. We assume $t>0$. Since

$$\rho^{-1}(\chi_{t[\beta]})=\rho^{-1}\Big(\sum_{\substack{\alpha,\beta\\ |\alpha|=|\sigma\beta|=t+|\beta|}}\chi_{[\alpha.\sigma\beta]}\Big)=\sum_{\substack{\alpha,\beta\\ |\alpha|=|\sigma\beta|=t+|\beta|}}s_{\sigma\beta}p_{r(\alpha\sigma\beta)}s_{\sigma\beta}^*,$$

we have $\tau'(\rho^{-1}(\chi_{t[\beta]}))=\tau'\Big(\sum_{\substack{\alpha,\beta\\ |\alpha|=|\sigma\beta|=t+|\beta|}}p_{r(\alpha\sigma\beta)}\Big)=\tau'(p_{r(\beta)})$. This implies that

$$\tau'\circ\rho^{-1}(\chi_{t[\beta]})=\tau'\circ\rho^{-1}(\chi_{t+1[\beta]})=\tau'\circ\rho^{-1}(\chi_{t[\beta]}\circ T),$$

which can also be shown for $t\leq 0$ in a similar way. Thus $\tau'\circ\rho^{-1}$ is T-invariant because the span of functions $\chi_{t[\beta]}$ is dense in $C(\mathcal{O}_\omega)$. $\square$

Remarks 3.8. (1) Simplicity of $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ can also be shown by analyzing the path structure of the labeled space $(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$. For a labeled graph $(E,\mathcal{L})$, set

$$\overline{\mathcal{L}(E^\infty)}:=\{x\in\mathcal{A}^{\mathbb{N}}\mid x_{[1,n]}\in\mathcal{L}(E^n)\text{ for all }n\geq 1\}.$$

Then $\mathcal{L}(E^\infty)\subset\overline{\mathcal{L}(E^\infty)}$ is obvious, but it is possible to have $\mathcal{L}(E^\infty)\subsetneq\overline{\mathcal{L}(E^\infty)}$. For example, if $\omega=0^\infty.0101^201^301^4\cdots\in\{0,1\}^{\mathbb{Z}}$, then the path $1^\infty\in\overline{\mathcal{L}_\omega(E_{\mathbb{Z}})}$ does not appear as an infinite labeled path in $\mathcal{L}_\omega(E^\infty_{\mathbb{Z}})$. We say that a labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$

is *strongly cofinal* if for each $x \in \overline{\mathcal{L}(E^\infty)}$ and $[v]_l \in \overline{\mathcal{E}}$, there exist an $N \geq 1$ and a finite number of paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}(E^{\geq 1})$ such that

$$r(x_{[1,N]}) \subset \cup_{i=1}^m r([v]_l, \lambda_i).$$

This definition of strong cofinality is a modification of the definitions given in [4, 18] and the proof of [4, Theorem 6.4] can be slightly modified to prove that if $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal and disagreeable, the C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple. If ω is a sequence satisfying the assumption of this section, it is not hard to see that the labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is strongly cofinal. Then by Lemma 3.6, we know that $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is simple.

(2) In case ω is the Thue Morse sequence given in Example 2.7, one can directly show that the simple unital C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ admits a unique tracial state. Moreover, its exact values on typical elements of the form $s_\alpha p_A s_\beta^*$ can be obtained explicitly, which will be done in [22].

Remark 3.9. If (X, T) is a Cantor minimal system, T induces an automorphism T of $C(X)$,

$$T(f) = f \circ T^{-1}, \quad f \in C(X),$$

and it is well known that the crossed product $C(X) \times_T \mathbb{Z}$ is always simple (for example, see [11]). It is also known [15] that the crossed products $C(X) \times_T \mathbb{Z}$ are not AF because $K_1(C(X) \times_T \mathbb{Z}) = \mathbb{Z}$. But they are all $A\mathbb{T}$ algebras, hence finite algebras of stable rank one, and have real rank zero by [6]. Moreover their isomorphism classes are determined by the ordered K_0-groups

$$(K_0(C(X) \times_T \mathbb{Z}), K_0^+(C(X) \times_T \mathbb{Z}), [1]_0)$$

together with the distinguished order units $[1]_0$, where 1 is the unit projection of the crossed product.

If a Cantor minimal system $(\mathcal{O}_\omega, T)$ is uniquely ergodic, the following theorem implies together with Theorem 3.7 that the crossed product $C(\mathcal{O}_\omega) \times_T \mathbb{Z}$ has a unique tracial state, which is well known for uniquely ergodic minimal systems (X, T) of infinite spaces X (see [11, Corollary VIII.3.8]).

Theorem 3.10. *Let $\mathcal{A}$ be a finite alphabet with $|\mathcal{A}| \geq 2$, and let $\omega \in \mathcal{A}^{\mathbb{Z}}$ be a sequence such that the subshift $(\mathcal{O}_\omega, T)$ is a Cantor minimal system. Then there is an isomorphism*

$$\pi : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}}) \to C(\mathcal{O}_\omega) \times_T \mathbb{Z}$$

such that the restriction of π onto the fixed point algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ of the gauge action γ is an isomorphism onto $C(\mathcal{O}_\omega)$.

Proof. Proposition 3.2 (and its proof) says that the fixed point algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ is isomorphic to $C(\mathcal{O}_\omega)$ via the map ρ given by

$$\rho(s_\alpha p_{r(\beta\alpha)} s_\alpha^*) = \chi_{[\beta.\alpha]}, \quad \alpha, \beta \in \mathcal{L}_\omega^\sharp(E_{\mathbb{Z}}).$$

We show that there exists an isomorphism of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ onto the crossed product $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \rtimes_{T'} \mathbb{Z}$, where $T' := \rho^{-1} \circ T \circ \rho$ is the automorphism of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$.

Note first that T' satisfies the following

$$T'(s_\alpha p_{r(\beta\alpha)} s_\alpha^*) = s_{\alpha_2\cdots\alpha_n} p_{r(\beta\alpha)} s_{\alpha_2\cdots\alpha_n}^* \tag{12}$$

for $\alpha, \beta \in \mathcal{L}_\omega^\sharp(E_{\mathbb{Z}})$. In fact, $\rho\big(T'(s_\alpha p_{r(\beta\alpha)} s_\alpha^*)\big) = T(\rho(s_\alpha p_{r(\beta\alpha)} s_\alpha^*)) = T(\chi_{[\beta.\alpha]}) = \chi_{T([\beta.\alpha])} = \chi_{[\beta\alpha_1.\alpha_2\cdots\alpha_n]} = \rho\big(s_{\alpha_2\cdots\alpha_n} p_{r(\beta\alpha)} s_{\alpha_2\cdots\alpha_n}^*\big)$ where $n := |\alpha|$. With the unitary u implementing the automorphism T' (namely, $T' = Adu$), this can be written as

$$T'(s_\alpha p_{r(\beta\alpha)} s_\alpha^*) = u(s_\alpha p_{r(\beta\alpha)} s_\alpha^*)u^* = s_{\alpha_2\cdots\alpha_n} p_{r(\beta\alpha)} s_{\alpha_2\cdots\alpha_n}^*.$$

Particularly,

$$u p_{r(\beta)} u^* = u\big(\sum_{a\in\mathcal{A}} s_a p_{r(\beta a)} s_a^*\big)u^* = \sum_{a\in\mathcal{A}} p_{r(\beta a)} \tag{13}$$

holds. To find a desired isomorphism, we will find a representation of the labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ in the crossed product, and then apply the universal property of the C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$. We actually show that the following partial isometries

$$t_a := u^* p_{r(a)}, \quad a \in \mathcal{A}$$

in the crossed products $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \times_{T'} \mathbb{Z}$ form a representation of $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ together with the family of projections $\{p_A : A \in \overline{\mathcal{E}}_{\mathbb{Z}})$. By (13), $t_a^* t_a = p_{r(a)}$ and $t_a^* t_b = \delta_{a,b} p_{r(a)}$ are immediate for $a, b \in \mathcal{A}$. We also have

$$\begin{aligned} p_{r(\beta)} t_a &= p_{r(\beta)} u^* p_{r(a)} = u^*\big(\sum_{b\in\mathcal{A}} p_{r(\beta b)}\big) p_{r(a)} \\ &= u^* p_{r(\beta a)} = u^* p_{r(a)} p_{r(\beta a)} = t_a p_{r(\beta a)} \\ &= t_a p_{r(r(\beta),a)}. \end{aligned}$$

Since every $A \in \overline{\mathcal{E}}_{\mathbb{Z}}$ can be written as a finite union of generalized vertices (by Proposition 2.2) and a generalized vertex $[v]_l$ is clearly equal to a range $r(\beta)$ of $\beta \in \mathcal{L}_\omega(E^l v)$, we know that the above equalities hold for any $A \in \overline{\mathcal{E}}_{\mathbb{Z}}$. Finally we have to check

$$p_{r(\beta)} = \sum_{a\in\mathcal{A}} t_a p_{r(\beta a)} t_a^*,$$

but this follows directly from the definition of t_a and (13). Thus $\{t_a, p_A\}$ forms a representation of the labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ in the C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \rtimes_{T'} \mathbb{Z}$, and hence there exists a homomorphism

$$\pi : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}}) \to C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \rtimes_{T'} \mathbb{Z}$$

such that $\pi(s_a) = t_a$ and $\pi(p_A) = p_A$ ($a \in \mathcal{A}$, $A \in \overline{\mathcal{E}}_{\mathbb{Z}}$). The homomorphism π is injective since $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is simple by Theorem 3.7, and is surjective since $u^* = u^*(\sum_{a\in\mathcal{A}} p_{r(a)}) = \sum_{a\in\mathcal{A}} t_a \in \mathrm{Im}(\pi)$ and $s_\alpha p_{r(\beta\alpha)} s_\alpha^* = (u^*)^{|\alpha|} p_{r(\beta\alpha)} u^{|\alpha|} \in \mathrm{Im}(\pi)$ for all generators $s_\alpha p_{r(\beta\alpha)} s_\alpha^*$ of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$.

For the last assertion, it is enough to see that for $\alpha, \beta \in \mathcal{L}_\omega^\sharp(E_{\mathbb{Z}})$,

$$\pi(s_\alpha p_{r(\beta\alpha)} s_\alpha^*) = s_\alpha p_{r(\beta\alpha)} s_\alpha^*.$$

If $a \in \mathcal{A}$, then $\pi(p_{r(a)}) = \pi(s_a^* s_a) = t_a^* t_a = p_{r(a)} u u^* p_{r(a)} = p_{r(a)}$, and hence $\pi(p_{r(\alpha)}) = p_{r(\alpha)}$ holds for all $\alpha \in \mathcal{L}^\sharp_\omega(E_\mathbb{Z})$. The equality (12) shows that the inverse $(T')^{-1}$ of the automorphism T' on $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})^\gamma$ maps a projections $p_{r(\alpha)}$ to the projection $s_a p_{r(\alpha)} s_a^*$, where $a \in \mathcal{A}$ is the last letter of α. (If $\alpha = \epsilon$ is the empty word, $p_{r(\epsilon)} = s_\epsilon$ is the unit of the labeled graph C^*-algebra, hence $(T')^{-1}(p_{r(\epsilon)}) = p_{r(\epsilon)} = s_\epsilon p_{r(\epsilon)} s_\epsilon^*$ also holds.) Then for $\alpha = \alpha' a$ with $\alpha' \in \mathcal{L}^\sharp_\omega(E_\mathbb{Z})$, $a \in \mathcal{A}$, we have

$$\pi(s_a p_{r(\alpha)} s_a^*) = t_a p_{r(\alpha)} t_a^* = u^* p_{r(\alpha)} u = (T')^{-1}(p_{r(\alpha)}) = s_a p_{r(\alpha)} s_a^*$$

as desired. □

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